

# Bubbling 1/4 BPS solutions in type IIB and supergravity reductions on $S^n \times S^n$

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## Abstract

We extend the construction of bubbling 1/2 BPS solutions of Lin, Lunin and Maldacena (hep-th/0409174) in two directions. First we enquire whether bubbling 1/2 BPS solutions can be constructed in minimal 6d supergravity and second we construct solutions that are 1/4 BPS in type IIB. We find that the  $S^1 \times S^1$  bosonic reduction of (1,0) 6d supergravity to 4d gravity coupled to 2 scalars and a gauge field is consistent only provided that the gauge field obeys a constraint ( $F \wedge F = 0$ ). This is to be contrasted to the case of the  $S^3 \times S^3$  bosonic reduction of type IIB supergravity to 4d gravity, 2 scalars and a gauge field, where consistency is achieved without imposing any such constraints. Therefore, in the case of (1,0) 6d supergravity we are able to construct 1/2 BPS solutions, similar to those derived in type IIB, provided that this additional constraint is satisfied. This ultimately prohibits the construction of a family of 1/2 BPS solutions corresponding to a bubbling  $AdS_3 \times S^3$  geometry. Returning to type IIB solutions, by turning on an axion-dilaton field we construct a family of bubbling 1/4 BPS solutions. This corresponds to the inclusion of back-reacted D7 branes to the solutions of Lin, Lunin and Maldacena.

# 1 Introduction and Summary

A recent paper by Lin, Lunin and Maldacena [1] provided a nice supergravity realization of chiral primary operators in  $\mathcal{N} = 4$  super-Yang-Mills. These operators, with conformal dimension  $\Delta$  equal their  $U(1)_R$  charge, form a decoupled sector of BPS states which can be identified with a gauged quantum mechanics matrix model, with a harmonic oscillator potential [2, 3]. By going to the eigenvalue basis, the path integral measure acquires a Van der Monde determinant factor which makes the eigenvalues behave as fermions which fill the energy levels inside a harmonic potential well. There is yet another perspective on the dynamics of eigenvalues: they correspond to the electrons in a magnetic field which fill the lowest Landau levels; by dropping the kinetic term, the positions of the electrons in the plane become non-commutative/canonical conjugates (quantum Hall effect) [1, 4].

There appears to be a one-to-one correspondence between the phase space regions occupied by the eigenvalues, and a similar picture that characterizes the supergravity solutions. Specifically, the 1/2 BPS family of solutions of [1] is constructed in terms of an auxiliary function; the boundary conditions which must be enforced on this auxiliary function in order for the supergravity solution to be non-singular reproduce precisely the phase space configuration of the eigenvalues. On the supergravity side, the incompressibility of the “phase space” has to be tied to the requirement that the 5-form RR-flux be fixed. In particular, the ground state on the matrix model side is a circular quantum Hall droplet in phase space, while on the supergravity side, the same droplet corresponds to the  $\text{AdS}_5 \times S^5$  ground state, with the radius of the droplet related to the  $R_{\text{AdS}_5}^2$  radius. Small excitations of the Fermi sea on the matrix side correspond to  $\text{AdS}_5 \times S^5$  perturbations by gravitons (ripples on the ground state droplet) or giant gravitons (small holes inside the ground state droplet, or small droplets outside the ground state droplet) [1].

It is worth asking whether a similar picture might carry through for the case of  $\text{AdS}_3 \times S^3$  which is the near horizon geometry of a D1-D5 brane configuration, and whether there is a bubbling  $\text{AdS}_3$  configuration corresponding to perturbation by giant gravitons. The six-dimensional giant gravitons are also configurations with  $\Delta = J$ , but they have certain peculiar features: they exist only for a discrete set of values of the angular momentum ( $J = nN_5 + mN_1$ , where  $N_5, N_1$  are the numbers of  $D_5$ , respectively  $D_1$  branes), and the potential governing their size is flat. The dual gauge theory in this case is a 1+1 dimensional CFT living on the boundary of  $\text{AdS}_3$ .

To address this question we look for 1/2 BPS solutions to minimal six-dimensional supergravity, which have  $S^1 \times S^1$  isometry. More precisely we consider the following ansatz:

$$\begin{aligned} ds^2 &= g_{\mu\nu}(x)dx^\mu dx^\nu + e^{H(x)+G(x)}d\phi^2 + e^{H(x)-G(x)}d\tilde{\phi}^2, \\ -2H_{(3)} &= F_{(2)} \wedge d\phi + \tilde{F}_{(2)} \wedge d\tilde{\phi}, \end{aligned} \tag{1.1}$$

where  $x^\mu = \{t, y, x^1, x^2\}$ . This is a natural extension of the ansatz used in [1], and similarly reduces the problem to that of an effective four-dimensional theory. Requiring that this ansatz preserves supersymmetry, we found two possible sets of Killing spinors: one which is independent on the  $\phi, \tilde{\phi}$  coordinates, and which yields a conventional Kaluza-Klein dimensional reduction, and a second set which carries Kaluza-Klein momentum on the two circles. The latter set of Killing spinors gives rise to six-dimensional solutions which include the  $\text{AdS}_3 \times S^3$  and the maximally symmetric plane wave solutions, as well as the multi-center D1-D5 solutions. The metric and the self-dual 3-form are expressed in terms of the same auxiliary function as in the 10-dimensional case, namely  $\partial_{x_1}^2 z + \partial_{x_2}^2 z + y \partial_y (1/y \partial_y z) = 0$ . The corresponding metric is non-singular provided that the same boundary conditions as in [1] are imposed on the auxiliary function:  $z(x_1, x_2, y = 0) = \pm 1/2$ . However, this time the field equations are satisfied only if an additional constraint is enforced as well:  $F \wedge F = 0$ , which can be easily seen by inspecting the  $\phi \tilde{\phi}$  component of Einstein equations. This constraint appears as a non-linear first order differential equation which the auxiliary function must satisfy as well:  $(\partial_{x_1} z)^2 + (\partial_{x_2} z)^2 + (\partial_y z)^2 = (1 - 4z^2)^2 / (4y^2)$ . One can check that for the case of  $\text{AdS}_3 \times S^3$  and that of the maximally symmetric plane wave the corresponding auxiliary functions do satisfy this additional constraint. However, the image of a bubbling  $\text{AdS}_3$  appears to be incompatible with the additional constraint, and we have to conclude that the starting ansatz is too restrictive to describe giant gravitons on  $\text{AdS}_3 \times S^3$ . In fact, therein lies the resolution: in order to eliminate this constraint, we must allow for (at least) an off-diagonal metric component  $g_{\phi \tilde{\phi}} \neq 0$  in the ansatz. This would correspond in the 4-dimensional reduction language to keeping a non-vanishing axion field besides the two scalars and the gauge field. We hope to report more on this in a future work. We also considered the case when a tensor multiplet (dilaton, dilatino and anti-selfdual tensor field) is coupled to six-dimensional minimal supergravity. With the same metric ansatz as before, we arrived yet again at the same auxiliary function  $z(x^1, x^2, y)$ , same metric and the same constraint  $F \wedge F = 0$ .

Finally, we return to the 10-dimensional type IIB supergravity to investigate a family of 1/4 BPS solutions which preserve the same set of  $SO(4) \times SO(4)$  isometries as in [1] and which correspond to turning on an axion-dilaton field  $\tau$ , or equivalently, including the back-reaction of a stack of  $N_f$  D7 branes. We found it possible to embed the D7 branes in a way compatible with the metric ansatz of [1], provided that they are transverse to  $x^1, x^2$ . The presence of the D7 branes manifests in the metric by curving the  $x^1, x^2$  directions with an additional factor  $e^{\psi(x^1, x^2)} \propto \Im \tau$ . The auxiliary function  $z(x^1, x^2, y)$  obeys a slightly modified second order differential equation, but one still imposes the same boundary conditions at  $y = 0$ . By setting up a perturbative expansion in  $N_f$ , we notice that to first order in  $N_f$ , and near the D7 branes, their effect on the geometry is to create a deficit angle in the  $x^1, x^2$

plane. Given that the fluctuations in the  $Z = x^1 + ix^2$  direction transverse to the stack of D3 branes correspond to the BPS chiral operators that are singled out in the gauged quantum mechanics matrix model, and that their eigenvalues correspond to the electrons in the quantum Hall effect, we argue that the deficit angle in the supergravity  $x^1, x^2$  “phase space” can be interpreted as the fractional statistics of the electrons in a fractional quantum Hall system.

It is worth noting that, despite recent advances, the complete classification of supersymmetric backgrounds with fluxes remains a technically challenging issue, especially in higher dimensions or with a large number of supersymmetries. The analysis of [1] is striking in this regards, in that by postulating a minimal set of isometries, the problem of classification may be greatly simplified. In particular, by reducing a  $(4 + 2n)$ -dimensional system on  $S^n \times S^n$ , we end up having to work with only two scalars and a gauge field in four dimensions. Of course, it is tempting to view this reduction as a Kaluza-Klein reduction on  $S^n \times S^n$ . In general, care must be taken when introducing states in the massive Kaluza-Klein tower. Here, however, by truncating to the singlet sector on  $S^n \times S^n$ , one is able to obtain a consistent *bosonic* truncation. This is sufficient for investigating the supersymmetry of the original system, provided the full supersymmetry transformations are used. One unusual feature of this analysis is the possibility of obtaining bosonic backgrounds which do not appear supersymmetric in four dimensions, but which are nevertheless supersymmetric when viewed as a solution of the original higher dimensional theory. This phenomenon was noted in [5], where it was referred to as ‘supersymmetry without supersymmetry’.

Furthermore, this method of obtaining gravitational solutions by solving a harmonic equation in an auxiliary space is similar in spirit to the work of Weyl [6] (see also [7] for a generalization to higher dimensions), who found all static axisymmetric solutions to the four-dimensional vacuum Einstein equations by mapping the problem to a cylindrically symmetric Laplacian problem in an auxiliary flat three-dimensional space. What makes the problem tractable in the Weyl case is the presence of a sufficient number of commuting Killing symmetries. Of course, one of the interesting features of the Weyl solutions is that they represent non-supersymmetric configurations of black holes held together by rods or struts. Thus, in that case, it is rather surprising that they may be described by solutions of a harmonic equation. For BPS configurations this is somewhat less of a surprise, as they are expected to obey the principle of linear superposition. Nevertheless, it is suggestive that new results in the classification of supersymmetric vacua may be obtained by revisiting some of the Weyl solution techniques in the present context. In this sense, it may also be worth looking at the M-theory compactification of [1] on  $S^2 \times S^5$  from the four-dimensional perspective.

The paper is structured as follows: Section 2 is dedicated to a proof that the bosonic

reduction of type IIB supergravity on  $S^3 \times S^3$  by retaining the breathing modes of the two 3-spheres and a gauge field in the reduced 4-dimensional theory is a consistent *bosonic* reduction. We also review the supersymmetry analysis of [1], using a slightly different representation of the 10-dimensional Clifford algebra which allows for more streamlined expressions. Section 3, which is organized in the same fashion as Section 2, contains the analysis of the six-dimensional supergravity reduction on  $S^1 \times S^1$  to the same set of 4-dimensional fields, and arrives at the conclusion that the reduction is a consistent *bosonic* reduction provided that the constraint  $F \wedge F = 0$  is satisfied. In Section 4 we construct the six-dimensional solutions which are compatible with supersymmetry. The additional constraint does not allow for solutions corresponding to bubbling  $\text{AdS}_3 \times S_3$ . Section 5 details the construction of the 1/4 BPS family of solutions corresponding to a bubbling  $\text{AdS}_5 \times S^5$  in the presence of D7 branes.

We have included the most technical parts of our investigation in a set of appendices. Appendix A contains a unified treatment of both the type IIB and six-dimensional supergravity bosonic reductions on  $S^n \times S^n$ , where  $n = 3(1)$  for the 10(6)-dimensional case respectively. Appendix B discusses the integrability condition of the supersymmetry variations of both type IIB and six-dimensional supergravity, and highlights the difference between the two, in the sense that the constraint  $F \wedge F = 0$  shows up in all  $S^n \times S^n$  reduction cases other than  $n = 3$ . Finally, Appendix C contains the full set of differential identities for spinor bilinears implied by supersymmetry.

## 2 $S^3 \times S^3$ compactification of IIB supergravity

The bosonic fields of IIB supergravity are given by the NSNS fields  $g_{MN}$ ,  $B_{MN}$  and  $\phi$  as well as the RR field strengths  $F_{(1)}$ ,  $F_{(3)}$  and  $F_{(5)}^+$ . In the Einstein frame, the IIB action has the form

$$e^{-1}\mathcal{L} = R - \frac{\partial_M \tau \partial^M \bar{\tau}}{2(\Im \tau)^2} - \frac{G_{(3)} \cdot \bar{G}_{(3)}}{2 \cdot 3! \Im \tau} - \frac{\tilde{F}_{(5)}^2}{4 \cdot 5!} + \frac{1}{4i} \frac{C_{(4)} \wedge G_{(3)} \wedge \bar{G}_{(3)}}{\Im \tau}, \quad (2.1)$$

where the self-duality  $\tilde{F}_{(5)} = *\tilde{F}_{(5)}$  must still be imposed by hand on the equations of motion. Here the field strengths are defined by

$$\begin{aligned} G_{(3)} &= F_{(3)} - \tau H_{(3)}, \\ F_{(3)} &= dC_{(2)}, \quad H_{(3)} = dB_{(2)}, \\ \tilde{F}_{(5)} &= dC_{(4)} - \frac{1}{2}C_{(2)} \wedge H_{(3)} + \frac{1}{2}B_{(2)} \wedge F_{(3)}, \end{aligned} \quad (2.2)$$

and  $\tau = C_{(0)} + ie^{-\phi}$  is the familiar axion-dilaton.

Although we are not directly concerned with the entire fermionic sector, since we are interested in the Killing spinor equations, we will need the IIB gravitino and dilatino variations

$$\begin{aligned}\delta\Psi_M &= [D_M + \frac{i}{16 \cdot 5!} \tilde{F}_{NPQRS} \Gamma^{NPQRS} \Gamma_M] \varepsilon \\ &\quad - \frac{1}{96} (\Gamma_M \Gamma^{NPQ} + 2 \Gamma^{NPQ} \Gamma_M) G_{NPQ} \varepsilon^*, \\ \delta\lambda &= i \Gamma^M P_M \varepsilon^* - \frac{i}{24} G_{MNP} \Gamma^{MNP} \varepsilon.\end{aligned}\tag{2.3}$$

Here  $D_M = \nabla_M - \frac{i}{2} Q_M$  where  $P_M$  and  $Q_M$  are the scalar kinetic and composite  $U(1)$  connection, respectively.

Note that the NSNS sector of the IIB model can be reduced on  $S^3 \times S^3$  [8, 9] to yield the gauged  $N = 4$  Freedman-Schwarz theory [10]. What we are interested in at present, however, is a reduction with additional degrees of freedom, in particular the self-dual 5-form as well as metric breathing modes. Here we note that, from a Kaluza-Klein point of view, the breathing modes are actually part of the massive Kaluza-Klein tower. In general, in the Freedman-Schwarz supergravity context, massive Kaluza-Klein supermultiplets necessarily involve charged modes on the spheres. As a result, it would be inconsistent to retain a single massive multiplet without retaining the entire tower.

Nevertheless, it is always possible to obtain a consistent bosonic breathing mode reduction by retaining only singlets on  $S^3 \times S^3$  [11, 12]. While the truncated theory is non-supersymmetric, we may still explore the original ten-dimensional Killing spinor equations obtained from (2.3), even in the context of the reduced bosonic fields. In this fashion, bosonic solutions of the compactified theory may be lifted to supersymmetric backgrounds of the original IIB theory, so long as the original Killing spinor equations are satisfied.

We now follow [1], and turn to the sector where only  $\tilde{F}_{(5)}$  is turned on (in addition to the metric). In this case, the equations of motion obtained from (2.1) admit a consistent truncation, so the relevant ten-dimensional Lagrangian is of the form

$$e^{-1} \mathcal{L} = R - \frac{1}{4 \cdot 5!} F_{(5)}^2,\tag{2.4}$$

where  $F_{(5)} = dC_{(4)}$  and  $F_{(5)} = *F_{(5)}$  is to be imposed on the equations of motion. This system is now that of a single self-dual form-field coupled to gravity, admitting a straightforward reduction on  $S^3 \times S^3$ . In particular, we take a reduction ansatz preserving an  $SO(4) \times SO(4)$  isometry of the form

$$\begin{aligned}ds_{10}^2 &= g_{\mu\nu}(x) dx^\mu dx^\nu + e^{H(x)} (e^{G(x)} d\Omega_3^2 + e^{-G(x)} d\tilde{\Omega}_3^2), \\ F_{(5)} &= F_{(2)} \wedge \omega_3 + \tilde{F}_{(2)} \wedge \tilde{\omega}_3,\end{aligned}\tag{2.5}$$

The details of this reduction are given in Appendix A. In the end, we obtain an effective four-dimensional Lagrangian of the form

$$e^{-1}\mathcal{L} = e^{3H}\left[R + \frac{15}{2}\partial H^2 - \frac{3}{2}\partial G^2 - \frac{1}{4}e^{-3(H+G)}F_{\mu\nu}^2 + 12e^{-H}\cosh G\right]. \quad (2.6)$$

At this point it is worth noting that the model of [1] may be extended, not just by turning on the axion-dilaton, but also by retaining the 3-form field-strength with an ansatz of the form

$$G_{(3)} = G_{(3)}(x) + a(x)\omega_3 + \tilde{a}(x)\tilde{\omega}_3. \quad (2.7)$$

While this may be of interest for obtaining additional supersymmetric backgrounds, we will not further pursue this direction at present.

## 2.1 Supersymmetry variations

We now turn to the reduction of the IIB supersymmetry variations, (2.3). At present, since we only turn on the metric and self-dual 5-form, the only non-trivial variation is that of the IIB gravitino, which has the form

$$\delta\psi_M = [\nabla_M + \frac{i}{16\cdot 5!}F_{NPQRS}\Gamma^{NPQRS}\Gamma_M]\varepsilon. \quad (2.8)$$

Writing this out in components, and using the reduction ansatz (2.5), we find

$$\begin{aligned} \delta\psi_\mu &= [\nabla_\mu - \frac{1}{16}e^{-\frac{3}{2}(H+G)}F_{\nu\lambda}\Gamma^{\nu\lambda}\Gamma^{(3)}\Gamma_\mu]\varepsilon, \\ \delta\psi_a &= [\hat{\nabla}_a + \frac{1}{4}\Gamma_a\Gamma^\mu\partial_\mu(H+G) - \frac{1}{16}e^{-\frac{3}{2}(H+G)}F_{\mu\nu}\Gamma^{\mu\nu}\Gamma^{(3)}\Gamma_a]\varepsilon, \\ \delta\psi_{\tilde{a}} &= [\hat{\nabla}_{\tilde{a}} + \frac{1}{4}\Gamma_{\tilde{a}}\Gamma^\mu\partial_\mu(H-G) - \frac{1}{16}e^{-\frac{3}{2}(H+G)}F_{\mu\nu}\Gamma^{\mu\nu}\Gamma^{(3)}\Gamma_{\tilde{a}}]\varepsilon, \end{aligned} \quad (2.9)$$

where  $\Gamma^{(3)} = -\frac{i}{6}\epsilon_{abc}\Gamma^{abc}$ , and we have taken into account the chirality of IIB spinors,  $\Gamma^{11}\varepsilon = \varepsilon$ , where  $\Gamma^{11} = \frac{1}{10!}\epsilon_{M_1\cdots M_{10}}\Gamma^{M_1\cdots M_{10}}$ .

To proceed, we choose a Dirac decomposition respecting the  $S^3 \times S^3$  symmetry

$$\begin{aligned} \Gamma_\mu &= \gamma_\mu \times 1 \times 1 \times \sigma_1, \\ \Gamma_a &= 1 \times \sigma_a \times 1 \times \sigma_2, \\ \Gamma_{\tilde{a}} &= \gamma_5 \times 1 \times \sigma_{\tilde{a}} \times \sigma_1. \end{aligned} \quad (2.10)$$

Here the  $\sigma$ 's are ordinary Pauli matrices, while  $\gamma_5 = \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^{\mu\nu\rho\sigma}$ . It is straightforward to see that, in this representation, the respective ‘chirality’ matrices are

$$\begin{aligned} \Gamma^{(5)} &= \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\Gamma^{\mu\nu\rho\sigma} = \gamma_5 \times 1 \times 1 \times 1, \\ \Gamma^{(3)} &= -\frac{i}{3!}\epsilon_{abc}\Gamma^{abc} = 1 \times 1 \times 1 \times \sigma_2, \\ \Gamma^{(\tilde{3})} &= -\frac{i}{3!}\epsilon_{\tilde{a}\tilde{b}\tilde{c}}\Gamma^{\tilde{a}\tilde{b}\tilde{c}} = \gamma_5 \times 1 \times 1 \times \sigma_1, \\ \Gamma^{11} &= \frac{1}{10!}\epsilon_{M_1\cdots M_{10}}\Gamma^{M_1\cdots M_{10}} = i\Gamma^{(5)}\Gamma^{(3)}\Gamma^{(\tilde{3})} = 1 \times 1 \times 1 \times \sigma_3. \end{aligned} \quad (2.11)$$

With this decomposition, the gravitino transformation becomes

$$\begin{aligned}
\delta\psi_\mu &= [\nabla_\mu + \frac{i}{16}e^{-\frac{3}{2}(H+G)}F_{\nu\lambda}\gamma^{\nu\lambda}\gamma_\mu]\epsilon, \\
\delta\psi_a &= \hat{\nabla}_a\epsilon - \frac{i}{4}\sigma_a[\gamma^\mu\partial_\mu(H+G) - \frac{i}{4}e^{-\frac{3}{2}(H+G)}F_{\mu\nu}\gamma^{\mu\nu}]\epsilon, \\
\delta\psi_{\tilde{a}} &= \hat{\nabla}_{\tilde{a}}\epsilon + \frac{1}{4}\gamma_5\sigma_{\tilde{a}}[\gamma^\mu\partial_\mu(H-G) + \frac{i}{4}e^{-\frac{3}{2}(H+G)}F_{\mu\nu}\gamma^{\mu\nu}]\epsilon.
\end{aligned} \tag{2.12}$$

We now write the complex IIB spinor as  $\varepsilon = \epsilon \times \chi \times \tilde{\chi}$  where  $\chi$  and  $\tilde{\chi}$  are Killing spinors on the respective unit three-spheres. Since they satisfy the Killing spinor equations

$$[\hat{\nabla}_a + \frac{i\eta}{2}\hat{\sigma}_a]\chi = 0, \quad [\hat{\nabla}_{\tilde{a}} + \frac{i\tilde{\eta}}{2}\hat{\sigma}_{\tilde{a}}]\tilde{\chi} = 0, \tag{2.13}$$

where  $\eta = \pm 1$  and  $\tilde{\eta} = \pm 1$ , the transformations (2.12) may be rewritten as

$$\delta\psi_\mu = [\nabla_\mu + \frac{i}{16}e^{-\frac{3}{2}(H+G)}F_{\nu\lambda}\gamma^{\nu\lambda}\gamma_\mu]\epsilon, \tag{2.14}$$

$$\delta\chi_H = [\gamma^\mu\partial_\mu H + e^{-\frac{1}{2}H}(\eta e^{-\frac{1}{2}G} - i\tilde{\eta}\gamma_5 e^{\frac{1}{2}G})]\epsilon, \tag{2.15}$$

$$\delta\chi_G = [\gamma^\mu\partial_\mu G - \frac{i}{4}e^{-\frac{3}{2}(H+G)}F_{\mu\nu}\gamma^{\mu\nu} + e^{-\frac{1}{2}H}(\eta e^{-\frac{1}{2}G} + i\tilde{\eta}\gamma_5 e^{\frac{1}{2}G})]\epsilon. \tag{2.16}$$

Here we have defined

$$\begin{aligned}
\psi_a &= -\frac{i}{4}\sigma_a(\chi_H + \chi_G), \\
\psi_{\tilde{a}} &= \frac{1}{4}\gamma_5\sigma_{\tilde{a}}(\chi_H - \chi_G).
\end{aligned} \tag{2.17}$$

To summarize, we have achieved a consistent bosonic breathing-mode reduction of the truncated IIB theory on  $S^3 \times S^3$ . The resulting four-dimensional Lagrangian is given by (2.6), while the reduction of the supersymmetry variations results in the system (2.14)–(2.16). Note that, while the variations have a typical form associated with four-dimensional  $\mathcal{N} = 2$  supergravity, they nevertheless should not be thought of as supersymmetries of the effective four-dimensional theory. This is because the bosonic truncation to singlets on  $S^3 \times S^3$  does not (and cannot) retain the complete supermultiplet content in the massive Kaluza-Klein sector, as indicated previously.

By reducing to an effective four-dimensional theory, we have ended up with a fairly simple system to investigate. Of course, we are mainly concerned with solving the Killing spinor equations derived from (2.14)–(2.16). In contrast to the approach of [1], we may now work directly in a four-dimensional context, even though the equations originated from the full IIB gravitino variation, (2.3). In addition, by reducing the six-dimensional  $\mathcal{N} = (1, 0)$  solution to four dimensions, we would similarly obtain an effective theory of the same general form as (2.6) and (2.14)–(2.16). Thus, using the four-dimensional picture, we will be able to solve both model simultaneously.



### 3 $S^1 \times S^1$ compactification of Minimal $D = 6$ Supergravity

We now turn to the case of minimal  $D = 6$ ,  $\mathcal{N} = (1, 0)$  supergravity which admits an  $\text{AdS}_3 \times S^3$  solution corresponding to the near horizon of the D1-D5 system. The field content of this theory is given by the gravity multiplet  $(g_{MN}, \psi_M, B_{MN}^+)$ , where  $B_{MN}^+$  denotes a two-form potential with self-dual field strength,  $H_{(3)} = *H_{(3)}$  where  $H_{(3)} = dB_{(2)}^+$ . This is often extended by the addition of a dilaton multiplet  $(B_{MN}^-, \lambda, \phi)$ , as the inclusion of both chiralities of  $B_{MN}$  then allow a covariant Lagrangian formulation. Furthermore, the Salam-Sezgin model [13] may be obtained by coupling to an Abelian vector multiplet  $(A_\mu, \chi)$ . In the following section, we will consider the addition of the dilaton multiplet. Here, however, we only focus on the minimal supergravity without dilaton or vector multiplet.

The bosonic sector of the minimal theory may be described by the Lagrangian

$$e^{-1}\mathcal{L} = R - \frac{1}{2 \cdot 3!}H_{(3)}^2 \quad (3.1)$$

where the self-duality condition on  $H_{(3)}$  remains to be imposed after deriving the equations of motion. Note that, for convenience, we have chosen to normalize  $H_{(3)}$  as if it were an unconstrained form-field. The resulting equations are simply

$$R_{MN} = \frac{1}{4}H_{MPQ}H_N{}^{PQ}, \quad H_{(3)} = *H_{(3)}, \quad dH_{(3)} = 0. \quad (3.2)$$

In addition, we note that the gravitino variation is given by

$$\delta\psi_M = [\nabla_M + \frac{1}{48}H_{NPQ}\Gamma^{NPQ}\Gamma_M]\epsilon. \quad (3.3)$$

These are the starting points of the reduction.

Analogous to the  $S^3 \times S^3$  reduction of IIB supergravity, we reduce the  $\mathcal{N} = (1, 0)$  theory on  $S^1 \times S^1$ . This is, of course, a familiar situation, as it is simply an ordinary Kaluza-Klein reduction on  $T^2$ , specialized to a rectangular torus. Before proceeding, it is worthwhile recalling the standard Kaluza-Klein result. Since this supergravity involves eight real supercharges, it reduces to a  $\mathcal{N} = 2$  theory in four dimensions. Since the six-dimensional metric reduces to a four-dimensional metric, two vectors and three scalars (two dilatonic and one axionic), and the self-dual  $B_{MN}^+$  reduces to a vector and axionic scalar, the resulting four-dimensional theory consists of  $\mathcal{N} = 2$  supergravity coupled to two vector multiplets.

In the present case, however, we specify a reduction ansatz of the form

$$\begin{aligned} ds_6^2 &= g_{\mu\nu}(x)dx^\mu dx^\nu + e^{H(x)}(e^{G(x)}d\phi^2 + e^{-G(x)}d\tilde{\phi}^2), \\ H_{(3)} &= -\frac{1}{2}F_{(2)} \wedge d\phi - \frac{1}{2}\tilde{F}_{(2)} \wedge d\tilde{\phi}, \end{aligned} \quad (3.4)$$

which preserves the  $SO(2) \times SO(2)$  isometry. The  $-1/2$  factors in the  $H_{(3)}$  ansatz are physically unimportant, but are chosen for later notational convenience in the reduced

supersymmetry variations. In the framework of a torus reduction, this ansatz corresponds to setting both metric gauge fields and both axionic scalars to zero. As a result, this ends up corresponding to an inconsistent truncation of the original  $\mathcal{N} = (1, 0)$  theory. Nevertheless, as we demonstrate below, this inconsistency manifests itself solely in one additional constraint that must be imposed by hand on the effective four-dimensional theory.

Given the reduction ansatz (3.4), it is straightforward to apply the results of Appendix A (while taking into account the normalization of  $F_{(2)}$ ) to arrive at the effective four-dimensional Lagrangian

$$e^{-1}\mathcal{L} = e^H[R + \tfrac{1}{2}\partial H^2 - \tfrac{1}{2}\partial G^2 - \tfrac{1}{8}e^{-(H+G)}F_{\mu\nu}^2], \quad (3.5)$$

where we note that  $\tilde{F}_{(2)} = -*_4 e^{-G}F_{(2)}$ . If desired,  $F_{(2)}$  may be canonically normalized by taking  $F_{(2)} \rightarrow \sqrt{2}F_{(2)}$ . However, it will be clear below when considering the supersymmetry variations why we avoid this last step. In addition, the reduction of the  $R_{\phi\tilde{\phi}}$  component of the Einstein equation results in a constraint  $F_{(2)} \wedge F_{(2)} = 0$ . This is the manifestation of the inconsistency in the reduction alluded to above. Ordinarily  $F_{(2)} \wedge F_{(2)}$  will source one of the axions. However, by truncating them away, we can no longer allow such a source. Nevertheless, so long as we satisfy this constraint, all solutions to the effective four-dimensional theory may be lifted to provide solutions to the original  $\mathcal{N} = (1, 0)$  model.

This reduction differs from the  $S^3 \times S^3$  case since, unlike the spheres, the circles are flat. Thus no potential is generated in the reduced theory. For the same reason, the scalar  $H$ , which plays the rôle of a breathing mode in the sphere reduction, is instead an ordinary massless scalar in the present case.

### 3.1 Supersymmetry variations

Having completed the reduction of the bosonic sector, we now proceed to reduce the gravitino variation, (3.3). Although the  $\mathcal{N} = (1, 0)$  theory is generally formulated with symplectic-Majorana Weyl spinors, here the six-dimensional spinors may be taken to be simply complex Weyl, satisfying the left-handed projection  $\Gamma^7\varepsilon = -\varepsilon$  as well as  $\Gamma^7\psi_M = -\psi_M$ , where  $\Gamma^7 = \frac{1}{6!}\epsilon_{M_1\dots M_6}\Gamma^{M_1\dots M_6}$ .

To obtain an effective four-dimensional description of the supersymmetry, we find it convenient to decompose the six-dimensional Dirac matrices according to

$$\begin{aligned} \Gamma_\mu &= \gamma_\mu \times \sigma_1, \\ \Gamma_4 &= 1 \times \sigma_2, \\ \Gamma_5 &= \gamma_5 \times \sigma_1. \end{aligned} \quad (3.6)$$

Indices 4 and 5 correspond to coordinates  $\phi$  and  $\tilde{\phi}$ , respectively. It is straightforward to see

that

$$\begin{aligned}\Gamma^{(5)} &= \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\Gamma^{\mu\nu\rho\sigma} = \gamma_5 \times 1, \\ \Gamma^7 &= \frac{1}{6!}\epsilon_{M_1\cdots M_6}\Gamma^{M_1\cdots M_6} = -1 \times \sigma_3.\end{aligned}\tag{3.7}$$

As a result, left-handed six-dimensional spinors may be written as  $\varepsilon = \epsilon \times \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ .

Noting that

$$H_{MNP}\Gamma^{MNP} = -\frac{3}{2}e^{-\frac{1}{2}(H+G)}F_{\mu\nu}\Gamma^{\mu\nu}\Gamma^4(1 + \Gamma^7),\tag{3.8}$$

[taking into account the unusual normalization of (3.4)] and using the above Dirac decomposition, the gravitino variation becomes

$$\begin{aligned}\delta\psi_\mu &= [\nabla_\mu + \frac{i}{16}e^{\frac{1}{2}(H+G)}F_{\nu\lambda}\gamma^{\nu\lambda}\gamma_\mu]\epsilon, \\ \delta\psi_\phi &= [\partial_\phi - \frac{i}{4}e^{\frac{1}{2}(H+G)}\gamma^\mu\partial_\mu(H+G) - \frac{1}{16}F_{\mu\nu}\gamma^{\mu\nu}]\epsilon, \\ \delta\psi_{\tilde{\phi}} &= [\partial_{\tilde{\phi}} + \frac{1}{4}e^{\frac{1}{2}(H-G)}\gamma^5\gamma^\mu\partial_\mu(H-G) + \frac{i}{16}e^{-G}F_{\mu\nu}\gamma^5\gamma^{\mu\nu}]\epsilon\end{aligned}\tag{3.9}$$

Up to this point, we have followed a conventional Kaluza-Klein reduction which involves truncation to zero-modes only on  $S^1 \times S^1$ . However, we are not necessarily interested in obtaining a consistent truncation to four dimensions, but rather wish to obtain supersymmetric configurations of the original  $\mathcal{N} = (1, 0)$  theory. We thus relax the Kaluza-Klein condition on the supersymmetry parameter  $\epsilon$ . In particular, we take  $\epsilon(x, \phi, \tilde{\phi}) = e^{i(q\phi + \tilde{q}\tilde{\phi})}\epsilon(x)$ , where the Kaluza-Klein momenta  $q$  and  $\tilde{q}$  are quantized in half-integer units (as appropriate for a spinor on a circle).

For convenience of notation, we redefine the (half-integer) Kaluza-Klein charges as  $q = -\eta/2$  and  $\tilde{q} = -\tilde{\eta}/2$ . Then, for spinors charged on the two circles, we may make a simple replacement

$$\partial_\phi \rightarrow -i\frac{\eta}{2}, \quad \partial_{\tilde{\phi}} \rightarrow -i\frac{\tilde{\eta}}{2}.\tag{3.10}$$

As a result, the above transformations may be rewritten as

$$\begin{aligned}\delta\psi_\mu &= [\nabla_\mu + \frac{i}{16}e^{-\frac{1}{2}(H+G)}F_{\nu\lambda}\gamma^{\nu\lambda}\gamma_\mu]\epsilon, \\ \delta\chi_H &= [\gamma^\mu\partial_\mu H + e^{-\frac{1}{2}H}(\eta e^{-\frac{1}{2}G} - i\tilde{\eta}\gamma_5 e^{\frac{1}{2}G})]\epsilon, \\ \delta\chi_G &= [\gamma^\mu\partial_\mu G - \frac{i}{4}e^{-\frac{1}{2}(H+G)}F_{\mu\nu}\gamma^{\mu\nu} + e^{-\frac{1}{2}H}(\eta e^{-\frac{1}{2}G} + i\tilde{\eta}\gamma_5 e^{\frac{1}{2}G})]\epsilon,\end{aligned}\tag{3.11}$$

where we have defined the linear combinations

$$\begin{aligned}\psi_\phi &= -\frac{i}{4}e^{\frac{1}{2}(H+G)}(\chi_H + \chi_G), \\ \psi_{\tilde{\phi}} &= \frac{1}{4}e^{\frac{1}{2}(H-G)}\gamma^5(\chi_H - \chi_G).\end{aligned}\tag{3.12}$$

In this form, the supersymmetry variations (3.11) resemble those of the  $S^3 \times S^3$  reduced IIB theory, (2.14)–(2.16). However, in this case, the parameters  $\eta$  and  $\tilde{\eta}$  may take on any

integer values include zero. Ordinary Killing spinors of the massless sector Kaluza-Klein reduction are obtained for  $\eta = \tilde{\eta} = 0$ , while charged Killing spinors are obtained otherwise. We keep in mind, however, that the bosonic sector is unchanged and always has the form of (3.5) regardless of the structure of the Killing spinors.

This reduction framework provides yet another example where the supersymmetry of spaces such as  $\text{AdS}_3 \times S^3$ , viewed as a fibration, involve Killing spinors charged along  $U(1)$  fibers. Proper supersymmetry counting then involves proper identification of the fiber  $U(1)$  charges [5, 14, 12].

## 4 Supersymmetry analysis

In the previous two sections, we have demonstrated that both the reduction of IIB theory on  $S^3 \times S^3$  and the reduction of  $\mathcal{N} = (1, 0)$  supergravity on  $S^1 \times S^1$  lead to similar effective four-dimensional descriptions. In particular, this similarity is evident not only in the bosonic sectors (2.6) and (3.5) but also in the supersymmetry transformations (2.14)–(2.16) and (3.11). At first sight, this is actually somewhat surprising, as the details of the fermionic sector, and in particular the mechanics of the supersymmetry algebra, are very much dimension dependent. However, given that both theories truncate to an identical field content, it is perhaps not unreasonable to expect that the effective four-dimensional descriptions of the supersymmetry transformations necessarily have a similar form.

In fact, comparing (2.14)–(2.16) with (3.11), we find that they may both be written in the form

$$\begin{aligned}\delta\psi_\mu &= [\nabla_\mu + \frac{i}{16}e^{-\frac{n}{2}(H+G)}F_{\nu\lambda}\gamma^{\nu\lambda}\gamma_\mu]\epsilon \\ \delta\chi_H &= [\gamma^\mu\partial_\mu H + e^{-\frac{1}{2}H}(\eta e^{-\frac{1}{2}G} - i\tilde{\eta}\gamma_5 e^{\frac{1}{2}G})]\epsilon, \\ \delta\chi_G &= [\gamma^\mu\partial_\mu G - \frac{i}{4}e^{-\frac{n}{2}(H+G)}F_{\mu\nu}\gamma^{\mu\nu} + e^{-\frac{1}{2}H}(\eta e^{-\frac{1}{2}G} + i\tilde{\eta}\gamma_5 e^{\frac{1}{2}G})]\epsilon,\end{aligned}\tag{4.1}$$

where  $n = 3$  or  $1$  corresponds to the  $S^3 \times S^3$  or  $S^1 \times S^1$  cases, respectively. This description of supersymmetry allows for a unified analysis of half-BPS solutions, using the methods of [1, 15–18]<sup>1</sup>. There are several differences to note about the two cases, however. Firstly, for  $n = 3$ , the choice of  $\eta = \pm 1$  and  $\tilde{\eta} = \pm 1$  is required based on satisfying the Killing spinor equations on the two 3-spheres. However, for  $n = 1$ , the parameters  $\eta$  and  $\tilde{\eta}$  refer to  $U(1)$  charges along the two circles, and may be chosen to be arbitrary integers including zero (to be consistent with charge quantization). Secondly, although  $F_{(2)}$  shows up identically in (4.1), there is actually a factor of two difference in the field strength terms in the bosonic Lagrangians, (2.6) and (3.5). This factor conspires with the  $n$ -dependence in the exponential

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<sup>1</sup>Note that general solutions of minimal  $\mathcal{N} = (1, 0)$  supergravity have been classified in [17]. Solutions to IIB theory on  $S^3 \times S^3$  were of course examined in [1].

prefactors  $e^{-\frac{n}{2}(H+G)}$  in (4.1) so as to yield the appropriate (distinct) bosonic equations of motion based on integrability of the supersymmetry variations. This issue of integrability is investigated in Appendix B. Finally, while not directly evident above, it is important to realize that when  $n = 1$  there is an added constraint,  $F_{(2)} \wedge F_{(2)} = 0$ , that must be checked for the solution.

Regardless of the differences mentioned above, we begin with a unified treatment of both cases. Our analysis parallels that of [1]. Thus we first assume that  $\epsilon$  is a Killing spinor and proceed by forming the spinor bilinears

$$\begin{aligned} f_1 &= \bar{\epsilon} \gamma^5 \epsilon, & f_2 &= i \bar{\epsilon} \epsilon, \\ K^\mu &= \bar{\epsilon} \gamma^\mu \epsilon, & L^\mu &= \bar{\epsilon} \gamma^\mu \gamma^5 \epsilon, \\ Y_{\mu\nu} &= i \bar{\epsilon} \gamma_{\mu\nu} \gamma^5 \epsilon. \end{aligned} \tag{4.2}$$

The factors of  $i$  are chosen to make these tensor quantities real. Then, by Fierz rearrangement, we may prove the algebraic identities

$$L^2 = -K^2 = f_1^2 + f_2^2, \quad K \cdot L = 0. \tag{4.3}$$

Next we turn to the differential identities. While the complete set of such identities are provided in Appendix C, only a subset suffices for the present analysis. Without yet making any assumptions about the metric, we may first fix the form of the scalar quantities  $f_1$  and  $f_2$ . Combining the differential identities for  $\nabla_\mu f_1$  and  $\nabla_\mu f_2$  in (C.1) with the  $L_\mu$  identities in (C.2) and (C.3), we obtain

$$\begin{aligned} \partial_\mu f_1 &= \frac{1}{4} e^{-\frac{n}{2}(H+G)} * F_{\mu\nu} K^\nu = \frac{1}{2} f_1 \partial_\mu (H - G), \\ \partial_\mu f_2 &= -\frac{1}{4} e^{-\frac{n}{2}(H+G)} F_{\mu\nu} K^\nu = \frac{1}{2} f_2 \partial_\mu (H + G), \end{aligned} \tag{4.4}$$

which may be integrated to obtain

$$f_1 = b e^{\frac{1}{2}(H-G)}, \quad f_2 = a e^{\frac{1}{2}(H+G)}. \tag{4.5}$$

The constants  $a$  and  $b$  are related through the identity  $\eta f_2 = -\tilde{\eta} e^G f_1$  of (C.2). In particular

$$a\eta + b\tilde{\eta} = 0. \tag{4.6}$$

Note that at this stage we still allow  $\eta$  and  $\tilde{\eta}$  to be arbitrary integers (including zero).

Having fixed the scalars  $f_1$  and  $f_2$ , we now turn to the vectors  $K_\mu$  and  $L_\mu$ . Here, we observe from (C.1) that the equations for  $K_\mu$  and  $L_\mu$  indicate that  $K_{(\mu;\nu)} = 0$  (so that  $K^\mu$  is a Killing vector) and  $dL = 0$ . As done in [1], this allows us to specialize the metric ansatz below. Before doing so, however, we note that these vectors are necessarily normalized by (4.3) to be

$$L^2 = -K^2 = f_1^2 + f_2^2 = e^H (a^2 e^G + b^2 e^{-G}). \tag{4.7}$$

In particular, this indicates that  $K^\mu$  is in fact a time-like Killing vector<sup>2</sup>. Furthermore, the  $L_\mu$  equations of (C.2) provide the constraints

$$\eta L_\mu = b \partial_\mu e^H, \quad \tilde{\eta} L_\mu = -a \partial_\mu e^H. \quad (4.8)$$

Following [1], we now choose a preferred coordinate basis so that the Killing vector  $K^\mu \partial_\mu$  corresponds to  $\partial/\partial t$  and the closed one-form  $L_\mu dx^\mu$  to  $dy$ , where  $t$  and  $y$  are two of the four coordinates. In particular, we write down the four-dimensional metric as

$$ds_4^2 = -h^{-2}(dt + V_i dx^i)^2 + h^2(dy^2 + \delta_{ij} dx^i dx^j), \quad (4.9)$$

where  $i, j = 1, 2$ . Note that we have already taken the spatial metric to be conformally flat based on the identical reasoning as [1]. The remaining components of the metric are  $V_i$ , to be determined below, and  $h^2$ , given from (4.7) to be

$$h^{-2} = e^H (a^2 e^G + b^2 e^{-G}). \quad (4.10)$$

In addition, for  $L = dy$ , (4.8) yields the constraints

$$\eta = b \partial_y e^H, \quad \tilde{\eta} = -a \partial_y e^H. \quad (4.11)$$

where we still allow for any of these constants  $\eta$ ,  $\tilde{\eta}$ ,  $a$  or  $b$  to be zero.

At this stage, we have sufficient information to fix the form of the field strengths  $F_{(2)}$  as well as  $dV$ . For  $F_{(2)}$ , we use the component relations

$$F_{\mu\nu} K^\nu = -\frac{4a}{n+1} \partial_\mu e^{\frac{n+1}{2}(H+G)}, \quad \tilde{F}_{\mu\nu} K^\nu = -\frac{4b}{n+1} \partial_\mu e^{\frac{n+1}{2}(H-G)}, \quad (4.12)$$

obtained from (4.4) as well as the explicit form of the metric (4.9) to find

$$\begin{aligned} F_{(2)} &= -\frac{4a}{n+1} d \left( e^{\frac{n+1}{2}(H+G)} \right) \wedge (dt + V) - \frac{4b}{n+1} h^2 e^{nG} *_3 d \left( e^{\frac{n+1}{2}(H-G)} \right), \\ \tilde{F}_{(2)} &= -\frac{4b}{n+1} d \left( e^{\frac{n+1}{2}(H-G)} \right) \wedge (dt + V) + \frac{4a}{n+1} h^2 e^{-nG} *_3 d \left( e^{\frac{n+1}{2}(H+G)} \right), \end{aligned} \quad (4.13)$$

where  $*_3$  denotes the Hodge dual with respect to the flat spatial metric. For  $dV$ , we take the antisymmetric part of  $\nabla_\mu K_\nu$  in (C.1), written in form notation as

$$dK = \frac{1}{2} e^{-\frac{n}{2}(H+G)} (f_2 F_{(2)} - f_1 * F_{(2)}), \quad (4.14)$$

and substitute in the expressions for the Killing vector  $K = -h^{-2}(dt + V)$  as well as for  $F_{(2)}$ . This gives both the known expression for  $h^{-2}$ , namely (4.10), as well as the relation

$$dV = -2abh^4 e^H *_3 dG. \quad (4.15)$$

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<sup>2</sup>It is useful to observe that this is in agreement with [17] where the 6-dimensional Killing vector  $\bar{\epsilon} \Gamma^M \epsilon$ ,  $M = 0, \dots, 5$  was shown to be null. The latter statement can be rephrased using our representation of 6-dimensional Dirac matrices as  $(\bar{\epsilon} \gamma^\mu \epsilon)^2 + f_1^2 + f_2^2 = 0$ .

## 4.1 Specialization of $\eta$ and $\tilde{\eta}$

Until now, we have allowed for all possible values of  $\eta$  and  $\tilde{\eta}$ . For sphere, as opposed to circle, compactifications ( $n > 1$ ) the only possible choices of  $\eta$  and  $\tilde{\eta}$  are  $\pm 1$ , as this is dictated by the Killing spinor equations on the sphere, (2.13). On the other hand, when  $n = 1$ , we may consider three distinct possibilities: both non-vanishing, one vanishing, and both vanishing.

### 4.1.1 Both $\eta$ and $\tilde{\eta}$ non-vanishing

We begin with the case of both  $\eta$  and  $\tilde{\eta}$  non-vanishing. In particular, to satisfy (4.6), we take  $a = -\tilde{\eta}$  and  $b = \eta$  with  $a^2 = b^2 = 1$ . The actual choice of  $a = \pm b$  corresponds to the case of chiral primaries with  $\Delta \mp J = 0$  in the dual field theory. For this case, (4.11) yields the simple result  $e^H = y$ , so that (4.10) becomes  $h^{-2} = 2y \cosh G$  [1]. As in [1], the consistency condition  $d^2V = 0$  following from (4.15) yields the second order differential equation  $d(y^{-1} *_3 dz) = 0$  or

$$(\partial_i^2 + y \partial_y \frac{1}{y} \partial_y) z = 0, \quad (4.16)$$

where  $z = \frac{1}{2} \tanh G$ .

Of course, it is not surprising that the analysis of [1] may be generalized away from  $n = 3$ , as the Killing spinor equations (4.1) have a relatively straightforward dependence on  $n$ . However, it is important to examine the complete consistency of the solution generated above, as in general solving the Killing spinor equations does not automatically yield a complete solution to the equations of motion, but only guarantees that a subset of the equations are solved. This issue of integrability is examined in detail in Appendix B. Here, it is sufficient to note that the  $F_{(2)}$  equations of motion are not obviously satisfied. Instead, by combining (4.13) with (4.15), we find that

$$dF_{(2)} = (n - 3) b h^2 y e^{\frac{n-1}{2}(H+G)} (dG \wedge *_3 dG - dH \wedge *_3 dH). \quad (4.17)$$

This demonstrates that, at least for  $n \neq 3$ , we must impose the additional constraint  $dG \wedge *_3 dG = dH \wedge *_3 dH$ , or

$$\partial_i z \partial_i z + \partial_y z \partial_y z = \frac{(1 - 4z^2)^2}{4y^2}. \quad (4.18)$$

That this constraint shows up for  $n \neq 3$  is related to the fact that the bosonic equations pick up a  $F_{(2)} \wedge F_{(2)} = 0$  constraint in this case as well. Unfortunately, this non-linear constraint is highly restrictive for functions  $z(x_1, x_2, y)$  already satisfying the Laplacian equation of motion (4.16). While we have not made an exhaustive search, we have only found the maximally symmetric  $\text{AdS}_3 \times S^3$  and plane-wave solutions to satisfy this constraint.

Without loss of generality, we choose  $a = b = 1$  for  $1/2$  BPS solutions. In this case, the generalization of [1] for  $n = 1$  as well as 3 may be summarized as follows:

$$ds_6^2 = -h^{-2}(dt + V_i dx^i)^2 + h^2(dy^2 + \delta_{ij} dx^i dx^j) + y(e^G d\Omega_n^2 + e^{-G} d\tilde{\Omega}_n^2), \quad (4.19)$$

$$F_{(2)} = -\frac{4}{n+1} \left[ d \left( y^{\frac{n+1}{2}} e^{\frac{n+1}{2}G} \right) \wedge (dt + V) + h^2 e^{nG} *_3 d \left( y^{\frac{n+1}{2}} e^{-\frac{n+1}{2}G} \right) \right], \quad (4.20)$$

where

$$h^{-2} = 2y \cosh G, \quad z \equiv \frac{1}{2} \tanh G, \quad dV = -\frac{1}{y} *_3 dz. \quad (4.21)$$

Note that  $F_{(2)}$  is only canonically normalized for  $n = 3$ . Furthermore, the function  $z$  must satisfy (4.16) for any  $n$ , and additionally the constraint (4.18) for  $n \neq 3$ .

#### 4.1.2 Only one of $\eta$ and $\tilde{\eta}$ non-vanishing

This and the subsequent possibility only applies when  $n = 1$ . Without loss of generality, we take  $\eta = 0$ ,  $\tilde{\eta} = \pm 1$ . In this case, the constraint (4.6) indicates that  $b = 0$ , so that in particular  $f_1 = 0$  or  $\bar{\epsilon}\gamma^5\epsilon = 0$ . To avoid the degenerate situation, we assume that  $a \neq 0$ . Taking  $a = -\tilde{\eta}$ , we see that (4.11) again gives  $e^H = y$ . This time, however, the relation (4.10) yields a single exponential,  $h^{-2} = ye^G$ . In addition, the field strength  $F_{(2)}$  is then given by (4.13)

$$F_{(2)} = -2a d(ye^G) \wedge (dt + V), \quad (4.22)$$

indicating that it is of pure electric type.

For both  $\eta$  and  $\tilde{\eta}$  non-vanishing, the second order equation giving the bubbling picture was obtained from the consistency condition  $d^2V = 0$ . Here, however,  $dV$  is trivially closed, as may be seen by setting  $b = 0$  in (4.15). Nevertheless, we must still satisfy the equation of motion for  $F_{(2)}$ , most conveniently expressed as  $d\tilde{F}_{(2)} = 0$  where

$$\tilde{F}_{(2)} = 2ay^{-1}e^{-2G} *_3 d(ye^G). \quad (4.23)$$

The resulting equation is simply  $d(y *_3 d(\frac{1}{y}e^{-G})) = 0$ , or

$$(\partial_i^2 + \frac{1}{y}\partial_y y \partial_y)\mathcal{H} = 0, \quad (4.24)$$

where  $\mathcal{H} = h^2 = \frac{1}{y}e^{-G}$  is a function of  $(x^1, x^2, y)$ . It is now evident that  $\mathcal{H}$  is a harmonic function in a four-dimensional space  $\mathbb{R}^2 \times \mathbb{R}^2$  where  $(x^1, x^2)$  span the first  $\mathbb{R}^2$  and  $y$  corresponds to the radial direction in the second (auxiliary)  $\mathbb{R}^2$ . Since there is no angular dependence in the second  $\mathbb{R}^2$ , the harmonic function is restricted to  $s$ -wave only solutions in the auxiliary space.

Putting together the above relations (and taking  $a = 1$ ), we find that the solution has the form

$$\begin{aligned} ds_6^2 &= -\mathcal{H}^{-1}(dt^2 + d\phi^2) + \mathcal{H}(\delta_{ij} dx^i dx^j + dy^2 + y^2 d\tilde{\phi}^2), \\ F_{(2)} &= 2dt \wedge d\frac{1}{\mathcal{H}}. \end{aligned} \quad (4.25)$$



Note that, because  $dV = 0$ , we have set  $V_i = 0$  since this may always be obtained by a suitable gauge transformation (diffeomorphism). It is now evident that we have reproduced the familiar multi-centered string solution in six-dimensions, restricted to singlet configurations along the  $\tilde{\phi}$  direction, under the assumption that the  $S^1$  parameterized by  $\phi$  has decompactified. This configuration arises naturally from the D1-D5 system with  $N_1 = N_5$ .

#### 4.1.3 Both $\eta$ and $\tilde{\eta}$ vanishing

Finally, for  $n = 1$ , we could have directly performed a standard Kaluza-Klein reduction on the circles, which would correspond to setting  $\eta = \tilde{\eta} = 0$ . Here, the constraint (4.6) becomes trivial, so that  $a$  and  $b$  may take on arbitrary values. Assuming at least one of the two is non-vanishing, then (4.11) implies that  $H$  is a constant, which we take to be zero. With this choice of  $H = 0$ , we then solve (4.10) for

$$h^{-2} = a^2 e^G + b^2 e^{-G}, \quad (4.26)$$

as well as (4.13) for

$$F_{(2)} = -2ae^G dG \wedge (dt + V) + 2bh^2 *_3 dG. \quad (4.27)$$

In addition, (4.15) gives

$$dV = -2abh^4 *_3 dG = -\frac{1}{ab} *_3 dz, \quad z \equiv \frac{1}{2} \frac{a^2 e^G - b^2 e^{-G}}{a^2 e^G + b^2 e^{-G}}, \quad (4.28)$$

provided  $ab \neq 0$ , or simply  $dV = 0$  otherwise.

For  $ab \neq 0$ , the consistency condition  $d^2 V = 0$  yields a three-dimensional Laplacian,  $d *_3 dz = 0$ , or

$$(\partial_i^2 + \partial_y^2)z = 0. \quad (4.29)$$

An additional constraint similar to (4.18), which arises from the  $F_{(2)}$  equation of motion, is still present. This time, however it simply states that  $dG \wedge *_3 dG = 0$  in the three-dimensional Euclidean space, so that  $G$  is necessarily a constant. As a result, we only obtain the Minkowski vacuum in this case.

For, say  $b = 0$ , on the other hand, the above relations reduce to

$$h^{-2} = a^2 e^G, \quad F_{(2)} = -2ade^G \wedge (dt + V), \quad dV = 0. \quad (4.30)$$

The equation of motion for  $F_{(2)}$  then gives  $d *_3 de^{-G} = 0$ , resulting in a solution of the form (setting  $a = 1$ )

$$\begin{aligned} ds_6^2 &= -e^G(dt^2 + d\phi^2) + e^{-G}(\delta_{ij}dx^i dx^j + dy^2 + d\tilde{\phi}^2), \\ F_{(2)} &= 2dt \wedge d\frac{1}{e^{-G}}. \end{aligned} \quad (4.31)$$

where  $\mathcal{H} = e^{-G}$  is harmonic in  $\mathbb{R}^3$  spanned by  $(x^1, x^2, y)$ . This solution is in fact of the same form as (4.25), and corresponds to a multi-centered string solution. This time, however, the Killing symmetry  $\partial/\partial\tilde{\phi}$  is not of an angular type, and both circles have decompactified. As a result, we have unfortunately been unable to obtain any new 1/2 BPS solutions of the minimal  $\mathcal{N} = (1, 0)$  system beyond the already familiar multi-centered string solutions.

## 4.2 $S^1 \times S^1$ reduction in the presence of a tensor multiplet

We now explore the possibility of evading the previous conclusion about the non-existence of a bubbling  $AdS_3 \times S^3$  solution, within the boundary of our ansatz, by enlarging the set of fields, from minimal 6-dimensional supergravity, to supergravity coupled to a tensor multiplet. The field content of the tensor multiplet is: dilaton  $\Phi$ , anti-selfdual tensor  $H_{\mu\nu\rho}^-$  and dilatino  $\lambda$ . The dilatino is Weyl, with opposite chirality than that of the gravitino.

We continue to work with the same metric ansatz as before, while the 3-form ansatz becomes

$$H_{(3)} = \left(F_{(2)} + K_{(2)}\right) \wedge d\phi_1 + \left(\tilde{F}_{(2)} + \tilde{K}_{(2)}\right) \wedge d\phi_2, \quad (4.32)$$

where  $\tilde{F}_{(2)} = -e^{-G} *_4 F_{(2)}$  and  $\tilde{K}_{(2)} = e^{-G} *_4 K_{(2)}$ . Thus  $F, \tilde{F}$  form the self-dual 3-form  $H_{(3)}^+$ , while  $K, \tilde{K}$  define the anti-selfdual 3-form  $H_{(3)}^-$ . The Bianchi identity and equation of motion read

$$dH_{(3)} = 0, \quad d\left(e^\Phi *_6 H_{(3)}\right) = 0. \quad (4.33)$$

Assuming that  $\Phi = \Phi(x)$ , in terms of the reduced form fields, these equations become

$$\begin{aligned} d(F_{(2)} + K_{(2)}) &= d(\tilde{F}_{(2)} + \tilde{K}_{(2)}) = 0, \\ d\Phi \wedge (F_{(2)} - K_{(2)}) + d(F_{(2)} - K_{(2)}) &= d\Phi \wedge (\tilde{F}_{(2)} - \tilde{K}_{(2)}) + d(\tilde{F}_{(2)} - \tilde{K}_{(2)}) = 0. \end{aligned} \quad (4.34)$$

The  $\phi_1\phi_2$  component of the Einstein equation yields the constraint

$$F_{(2)} \wedge F_{(2)} - K_{(2)} \wedge K_{(2)} = 0. \quad (4.35)$$

The supersymmetry variations of the supergravity multiplet are the same as before, (3.9), with the exception that  $F, \tilde{F}$  must be replaced by  $e^{\Phi/2}F, e^{\Phi/2}\tilde{F}$ . Correspondingly, the spinor bilinear equations (C.1)–(C.3) are modified by means of the same replacement. The immediate consequence of this observation is that we obtain as before  $f_2 = \exp((H + G)/2)$ ,  $f_1 = \exp((H - G)/2)$ ,  $K^\mu \partial_\mu$  is a Killing vector and  $L_\mu dx^\mu$  is still a closed form. Also, the relation  $h^{-2} = f_1^2 + f_2^2$  holds as well. Therefore the metric has become once more  $ds^2 = h^{-2}(dt + V)^2 + h^2(dy^2 + \sum_{i,j=1}^2 h_{ij} dx^i dx^j) + e^{H+G} d\phi_1^2 + e^{H-G} d\phi_2^2$ . We have learned also that

$$i_K F_{(2)} = -e^{H+G} e^{-\Phi/2} d(H + G), \quad i_K \tilde{F}_{(2)} = -e^{H-G} e^{-\Phi/2} d(H - G), \quad (4.36)$$

where  $i_K$  denotes the inner contraction with the Killing vector  $K = h^{-2}(dt + V)$ , and

$$dK = \frac{1}{2} \left( e^{H+G} e^{\Phi/2} F_{(2)} + e^{H-G} e^{\Phi/2} \tilde{F}_{(2)} \right). \quad (4.37)$$

Substituting  $K$  as well as (4.36) into the previous equation we find the same differential equation defining  $z = \frac{1}{2} \tanh(G)$  as (4.16). Thus the metric is identical to the one derived previously in the absence of the tensor multiplet.

Consistency of (4.37), namely  $d^2 K = 0$ , combined with Bianchi and equations of motion (4.34) leads to

$$K_{(2)} \wedge K_{(2)} = 0, \quad (4.38)$$

which in turn implies that  $F_{(2)} \wedge F_{(2)} = 0$ . We see that despite our efforts to avoid the  $F_{(2)} \wedge F_{(2)}$  constraint which translates into the additional non-linear differential equation that  $z(x_1, x_2, y)$  had to satisfy, we have to conclude that turning on the tensor multiplet did not achieve, as one might have hoped, a bubbling  $AdS_3 \times S^3$  picture. As mentioned in the introduction, a possible way to evade the negative conclusion on bubbling  $AdS_3 \times S^3$  solutions is to allow for a 4d axion field (arising from  $g_{\phi_1 \phi_2}$ ). In fact, a rather large class of supersymmetric 6d solutions of conical defect type [19] fall into this class of metrics.

## 5 Bubbling 1/4 BPS solutions: turning on an axion-dilaton

In this section we show how the 1/2 BPS family of solutions discovered recently by Lin, Lunin and Maldacena [1] can be modified to accommodate a holomorphic axion-dilaton field. Of course, in doing so we break the amount of supersymmetry that the new solutions preserve by half. We will end up with a family of 1/4 BPS solutions which have the same  $SO(4) \times SO(4)$  isometries inherited from the 1/2 BPS family.

Our interest in this class of 1/4 BPS solutions resides in its implications for the dual gauge theory. We expect that turning on the axion-dilaton field  $\tau$ , which amounts to adding D7 branes by appropriately including their back-reaction, will lead to the addition of flavor degrees of freedom to the dual gauge theory. By embedding  $N_f$  D7 branes in the initial  $AdS_5 \times S^5$  geometry, one adds  $N_f \mathcal{N} = 2$  hypermultiplets,  $Q$ , in the fundamental of  $N_c$  to the  $\mathcal{N} = 4$   $SU(N_c)$  dual gauge theory. The gauge theory superpotential is accordingly modified by the addition of the hypermultiplets to  $\text{Tr } X[Y, Z] + \bar{Q} Z Q$ .

More precisely, we begin our construction of 1/4 BPS solutions in type IIB supergravity with the following ansatz:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + e^{H+G} d\Omega_3^2 + e^{H-G} d\tilde{\Omega}_3^2, \quad (5.1)$$

$$F_{(5)} = F_{\mu\nu} dx^\mu \wedge dx^\nu \wedge d\Omega_3 + \tilde{F}_{\mu\nu} dx^\mu \wedge dx^\nu \wedge d\tilde{\Omega}_3, \quad (5.2)$$

$$\tau = \tau(x^1 + ix^2), \quad \text{with } x^\mu = \{t, y, x^1, x^2\}. \quad (5.3)$$

As in [20], we will be able to exploit the fact that the D3-D7 problem separates, with the D7 branes curving the space transverse to them, and the warping due to the D3 branes modified to accommodate the D7 branes' backreaction. The self-duality condition and the Bianchi identity of the 5-form imply for the reduced form fields

$$F = e^{3G} *_4 \tilde{F}, \quad F = dB, \quad \tilde{F} = d\tilde{B}. \quad (5.4)$$

Requiring that this solution is supersymmetric, we impose

$$\delta\psi_M = (\nabla_M - \frac{i}{2}Q_M)\varepsilon + \frac{i}{480}F_{M_1M_2M_3M_4M_5}\Gamma^{M_1M_2M_3M_4M_5}\varepsilon = 0 \quad (5.5)$$

$$\delta\lambda = iP_M\Gamma^M\varepsilon^* = 0, \quad (5.6)$$

where  $\psi_M$  and  $\lambda$  are the complex gravitino and dilatino, whose  $U(1)$  charges are  $1/2$  and  $3/2$  respectively. The axion and dilaton fields parameterize a scalar coset  $SL(2, R)/U(1)$ , with the  $U(1)$  connection given by

$$Q_M = -\frac{1}{2}\frac{\partial_M \Re\tau}{\Im\tau}, \quad (5.7)$$

and where  $g^{MN}P_MP_N^*$  represents the kinetic term of the sigma-model Lagrangian, with

$$P_M = -\frac{1}{2}\frac{\partial_M \Re\tau}{\Im\tau}. \quad (5.8)$$

Notice that the previous supersymmetry variations of the gravitino along the sphere directions, (2.15), (2.16) are not modified by the presence of the scalar fields, and that (2.14) contains a new term due to the  $Q$ -connection. The new constraint following from the supersymmetry variation of the dilatino only enforces

$$(\Gamma^1 + i\Gamma^2)\varepsilon = (\gamma^1 + i\gamma^2)\varepsilon = 0. \quad (5.9)$$

The spinor bilinear equations derived previously (C.1) are unchanged, because a bilinear of the type  $(\bar{\varepsilon}\dots\varepsilon)$  is  $U(1)$  neutral. However, the one-form

$$\omega = \epsilon^T C \gamma_\mu \epsilon dx^\mu, \quad (5.10)$$

where  $C$  is the charge conjugation matrix ( $\gamma^\mu = -C\gamma^{\mu,T}C$ ), is no longer closed as it was the case in the absence of the axion-dilaton field; rather it obeys

$$d\omega = iQ \wedge \omega. \quad (5.11)$$

Given that  $K_\mu$  is still a Killing vector and  $L_\mu dx^\mu$  is still an exact form, we can choose as before  $K^\mu$  as the generator of time translations,  $K^t = 1$ , and we choose  $L = \gamma dy$ , with  $\gamma = \pm 1$ . Therefore we arrive at the same form of the metric

$$ds_4^2 = -h^{-2}(dt + V_i dx^i)^2 + h^2(dy^2 + \tilde{h}_{ij} dx^i dx^j). \quad (5.12)$$

We can see now that the constraint (5.9) has become a projection condition on  $\epsilon$

$$(1 + i\gamma^{12})\epsilon = 0. \quad (5.13)$$

Using the Killing spinor equations we find the following set of equations

$$f_2 \partial_\mu H = -\tilde{\eta} e^{-(H-G)/2} L_\mu, \quad (5.14)$$

$$\partial_\mu f_2 = -e^{-3(H+G)/2} F_{\mu\nu} K^\nu = \frac{1}{2} f_2 \partial_\mu (H + G), \quad (5.15)$$

which allows the integration of both the spinor bilinear  $f_2$  and  $H$  as

$$f_2 = 4\alpha e^{(H+G)/2}, \quad B_t = -\alpha e^{2(H+G)}, \quad e^H = y, \quad (5.16)$$

where we fix the sign of  $\gamma$  such that  $-\tilde{\eta}\gamma = 1$ . Similarly we find

$$f_1 = 4\beta e^{(H-G)/2}, \quad \tilde{B}_t = -\beta e^{2(H-G)}. \quad (5.17)$$

With the choice  $4\beta = 1$ , we end up by fixing  $\alpha$  using on the one hand (4.10)

$$h^{-2} = f_1^2 + f_2^2, \quad (5.18)$$

and

$$\tilde{\eta} h^{-2} \partial_y e^H = \tilde{\eta} h^{-2} = \tilde{\eta} \left( \frac{f_2^2}{4\alpha} - \frac{\eta f_1^2}{4\tilde{\eta}\beta} \right), \quad (5.19)$$

on the other hand. The latter equation is obtained from (2.15) by multiplication with  $\bar{\epsilon}\gamma^5$ . We conclude that the last two equations imply  $4\alpha = 1$ , and  $\eta = -\tilde{\eta}$ . Substituting  $H$  into the Killing spinor equation (2.15) we identify another projector

$$\begin{aligned} 0 &= \left( \frac{1}{yh} \gamma^3 + \eta e^{-(H+G)/2} - i\tilde{\eta} \gamma_5 e^{-(H-G)/2} \right) \epsilon \\ &= \left( \sqrt{1 + e^{-2G}} \gamma^3 - \tilde{\eta} (i e^{-G} \gamma_5 + 1) \right) \epsilon. \end{aligned} \quad (5.20)$$

Moreover, using that

$$\begin{aligned} K^t &= h \bar{\epsilon} \gamma^0 \epsilon = h \epsilon^\dagger \epsilon = 1, \\ L_y &= h \bar{\epsilon} \gamma^3 \gamma_5 \epsilon = -\tilde{\eta}, \end{aligned} \quad (5.21)$$

one derives the last projector

$$(\tilde{\eta} - \gamma^0 \gamma^3 \gamma_5) \epsilon = (\tilde{\eta} - i\gamma^{12}) \epsilon = 0. \quad (5.22)$$

However, we do not have the freedom of two choices of sign for  $\eta$ , because of the first projection condition that we encountered (5.9) from the susy variation of the dilatino which identifies

$$\tilde{\eta} = -1. \quad (5.23)$$

Thus our solution ends up preserving only 1/4 of the 32 supersymmetries.

Finally, using the projectors we can solve for the Killing spinor

$$\epsilon = e^{(H+G)/4} \exp(i\delta\gamma_5\gamma_3)\epsilon_0, \quad \sinh\delta = e^{-G}, \quad \epsilon_0^\dagger\epsilon = 1. \quad (5.24)$$

Substituting the Killing spinor in (5.11), where still the only non-vanishing components are  $\omega_{\underline{1}}$  and  $\omega_{\underline{2}}$ , we realize that we end up with a conformally flat two dimensional space in the  $x^1, x^2$  directions:

$$\tilde{h}_{ij}dx^i dx^j = e^{\Psi(x^1, x^2)}((dx^1)^2 + (dx^2)^2), \quad (5.25)$$

$$\omega = e^{\Psi(x^1, x^2)}d(x_1 + i x_2). \quad (5.26)$$

Moreover, the conformal factor  $e^\Psi$  is related to the axion-dilaton field, because we argued earlier that the U(1) connection  $Q$  becomes the spin connection in this two dimensional space

$$\Psi(x_1, x_2) = \Im\tau(x_1 + i x_2). \quad (5.27)$$

In fact, there is even more freedom in defining  $\Psi(x_1, x_2)$  in terms of multiplication by a holomorphic and an antiholomorphic function  $\exp(\Psi) = \Im\tau \exp(f(x_1 + i x_2) + f^*(x_1 - i x_2))$ . This stems from the freedom of multiplying  $\epsilon_0$  by a phase:  $\exp((f - f^*)/2)\epsilon_0$ . Imposing modular invariance, with  $N_f$  D7 branes located at  $Z_i = (x_1 + i x_2)_i = 0$ , uniquely determines the conformal factor as

$$e^{\Psi(Z, Z^*)} = \Im\tau \left| \eta(\tau) \right|^4 \prod_{i=1}^{N_f} \left| Z - Z_i \right|^{-1/6}. \quad (5.28)$$

The corresponding geometry is non-singular provided that  $N_f < 24$ . Near the D7 branes, the axion-dilaton field behaves as a logarithm, and its equation of motion has delta-function singularities at the location of the D7 branes

$$\tau(Z) \approx \frac{i}{g} + \frac{1}{2\pi i} \sum_i \ln(Z - Z_i), \quad (5.29)$$

$$\exp(\Psi) \approx \frac{1}{g} - \frac{1}{2\pi} \sum_i \ln(|Z - Z_i|). \quad (5.30)$$

We are left only with determining the 5-form flux: given the similarity of our equations to those of [1], it is easy now to see that the bubbling 1/4 BPS solutions read

$$\begin{aligned} e^{-\Psi} \partial_i^2 z + y \partial_y \left( \frac{1}{y} \partial_y z \right) &= 0, \\ dV &= \frac{1}{y} *_3 dz, \quad z = \frac{1}{2} \tanh G, \\ F &= dB_t \wedge (dt + V) + B_t dV + d\hat{B}, \quad \tilde{F} = d\tilde{B}_t \wedge (dt + V) + \tilde{B}_t dV + d\hat{\tilde{B}}, \\ B_t &= \frac{1}{4} y^2 e^{2G}, \quad \tilde{B}_t = -\frac{1}{4} y^2 e^{-2G}, \\ d\hat{B} &= \frac{-1}{4} y^3 *_3 d \frac{z + 1/2}{y^2}, \quad d\hat{\tilde{B}} = \frac{-1}{4} y^3 *_3 d \frac{z - 1/2}{y^2}, \end{aligned} \quad (5.31)$$

where one should keep in mind that the Hodge symbol  $\ast_3$  in the three dimensional space is computed relative to the metric  $dy^2 + e^\Psi \sum_{i=1,2} (dx^i)^2$ .

To gauge the effect of the axion-dilaton field on the geometry, we can in a first order of approximation solve the differential equation which defines the auxiliary function  $z(x_1, x_2, y)$  perturbatively in  $N_f$ . We assume that all D7 branes are overlapping and we approximate the conformal factor by  $\Im\tau$  (5.30). We define polar coordinates in the  $(x^1, x^2)$  plane, and we redefine the radial coordinate  $\rho$  by  $r = \rho e^{\Psi/2}$ . The effect of this rescaling is to map the line element  $ds_2^2 = e^\Psi (d\rho^2 + \rho^2 d\varphi^2)$  into  $ds^2 \approx (1 - \frac{N_f}{2\pi})(dr^2 + r^2 d\varphi^2)$ . This is nothing else but a 2-dimensional space with a deficit angle. Therefore  $z(x_1, x_2, y)/y^2$  is to a first order of approximation still a harmonic function, but it is a harmonic function of a 6-dimensional space, with a deficit angle in the 2-plane defined by  $x_1, x_2$ .

Therefore the presence of the D7 branes, while not affecting to a dramatic degree the bubbling  $\text{AdS}_5 \times S^5$  picture, so that in particular it does not change the interpretation of  $z(x_1, x_2, y = 0) = \pm 1/2$  as a “phase space”, nevertheless induces a deficit angle in this plane. Since the fluctuations of the D3 branes in the direction  $Z = x_1 + ix_2$  become in the decoupling limit the BPS chiral primary operators defining the gauged quantum mechanics matrix model of [2, 3], a deficit angle in the  $(x_1, x_2)$  planes translates into a non-trivial monodromy of the chiral primary operators. This ultimately implies that the eigenvalues of  $Z$  have a non-trivial monodromy, or equivalently, the electrons participating in the quantum Hall effect (i.e. the eigenvalues) have fractional statistics.

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## A Bosonic reductions on $S^n \times S^n$

The reduction of the bosonic fields (metric and form-field) may be performed in arbitrary dimensions. For a reduction to four-dimensions on  $S^n \times S^n$ , we start with a  $D = 4 + 2n$  dimensional bosonic action of the form

$$e^{-1}\mathcal{L} = R - \frac{1}{2 \cdot (n+2)!} F_{(n+2)}^2. \quad (\text{A.1})$$

The resulting equations of motion are simply

$$R_{MN} = \frac{1}{2(n+1)!} \left[ (F^2)_{MN} - \frac{1}{2(n+2)} g_{MN} F^2 \right],$$

$$dF = d*F = 0. \quad (\text{A.2})$$

Note that here we have taken  $F_{(n+2)}$  to be canonically normalized. Furthermore, at this stage we do not impose self-duality on the form-field, although below we will show what modifications would be necessary to cover the self-dual case.

The reduction of the equations of motion, (A.2), proceeds by taking an ansatz of the form

$$\begin{aligned} ds^2 &= g_{\mu\nu}(x)dx^\mu dx^\nu + e^{H(x)} \left( e^{G(x)} d\Omega_n^2 + e^{-G(x)} d\tilde{\Omega}_n^2 \right), \\ F_{(n+2)} &= F_{(2)} \wedge \omega_n + \tilde{F}_{(2)}(x) \wedge \tilde{\omega}_n, \end{aligned} \quad (\text{A.3})$$

where  $\omega_n$  and  $\tilde{\omega}_n$  are volume forms on the respective unit  $n$ -spheres. Our goal is now to obtain the four-dimensional effective theory for the fields  $g_{\mu\nu}$ ,  $H$ ,  $G$ ,  $F_{(2)}$  and  $\tilde{F}_{(2)}$ .

At this point, it is worth recalling that, for sphere compactifications, the fields  $H$  and  $G$  are actually breathing mode scalars which live in the massive Kaluza-Klein tower. In general, it would be inconsistent to retain only a subset of the massive Kaluza-Klein states, as they typically source each other *ad infinitum*. However, the scalars  $H$  and  $G$  themselves are uncharged on the spheres, and hence this breathing mode reduction remains consistent by virtue of retaining only singlets on the spheres [11, 12].

We begin by reducing the form-field equation of motion. From the ansatz (A.3), we may obtain the Hodge dual

$$*F_{(n+2)} = *_4 e^{nG} \tilde{F}_{(2)} \wedge \omega_n + (-)^n *_4 e^{-nG} F_{(2)} \wedge \tilde{\omega}_n. \quad (\text{A.4})$$

At this point, we note that, since  $*_4 *_4 = -1$ , we may only impose a self-dual condition on  $F_{(n+2)}$  for odd  $n$  dimensions (*i.e.*  $D = 6$  or  $10$ ). In such dimensions, self-duality yields the relation  $\tilde{F}_{(2)} = - *_4 e^{-nG} F_{(2)}$ . In any case, we see that the  $F_{(n+2)}$  equation of motion reduces simply to

$$\begin{aligned} dF_{(2)} &= 0, & d(e^{-nG} *_4 F_{(2)}) &= 0, \\ d\tilde{F}_{(2)} &= 0, & d(e^{nG} *_4 \tilde{F}_{(2)}) &= 0. \end{aligned} \quad (\text{A.5})$$

Turning to the Einstein equation, we first compute the Ricci tensor for the metric ansatz (A.3)

$$\begin{aligned} R_{\mu\nu}^{(D)} &= R_{\mu\nu} - \frac{n}{2}(\partial_\mu H \partial_\nu H + \partial_\mu G \partial_\nu G) - n \nabla_\mu \nabla_\nu H, \\ R_{ab}^{(D)} &= \hat{R}_{ab} - \frac{n}{2} g_{ab} \partial^\mu H \partial_\mu (H + G) - \frac{1}{2} g_{ab} \square (H + G), \\ R_{\tilde{a}\tilde{b}}^{(D)} &= \hat{R}_{\tilde{a}\tilde{b}} - \frac{n}{2} g_{\tilde{a}\tilde{b}} \partial^\mu H \partial_\mu (H - G) - \frac{1}{2} g_{\tilde{a}\tilde{b}} \square (H - G). \end{aligned} \quad (\text{A.6})$$

Here  $\hat{R}_{ab} = (n-1)\hat{g}_{ab}$  and  $\hat{R}_{\tilde{a}\tilde{b}} = (n-1)\hat{g}_{\tilde{a}\tilde{b}}$  are the curvatures on the *unit*  $n$ -spheres with metrics  $\hat{g}_{ab}$  and  $\hat{g}_{\tilde{a}\tilde{b}}$ , respectively. The  $D$ -dimensional metric components in the sphere directions are  $g_{ab} = e^{H+G}\hat{g}_{ab}$  and  $g_{\tilde{a}\tilde{b}} = e^{H-G}\hat{g}_{\tilde{a}\tilde{b}}$ .



For the form-field, we compute

$$\begin{aligned}
F_{(n+2)}^2 &= \frac{1}{2}(n+2)! \left( e^{-n(H+G)} F^2 + e^{-n(H-G)} \tilde{F}^2 \right), \\
(F_{(n+2)}^2)_{\mu\nu} &= (n+1)! \left( e^{-n(H+G)} (F^2)_{\mu\nu} + e^{-n(H-G)} (\tilde{F}^2)_{\mu\nu} \right), \\
(F_{(n+2)}^2)_{ab} &= \frac{1}{2}(n+1)! e^{-n(H+G)} g_{ab} F^2, \\
(F_{(n+2)}^2)_{\tilde{a}\tilde{b}} &= \frac{1}{2}(n+1)! e^{-n(H-G)} g_{\tilde{a}\tilde{b}} \tilde{F}^2, \\
(F_{(3)}^2)_{a\tilde{a}} &= \hat{e}_a \hat{e}_{\tilde{a}} F_{\mu\nu} \tilde{F}^{\mu\nu} \quad \text{for } n=1.
\end{aligned} \tag{A.7}$$

These expressions allow us to work out the source for the Einstein equations. They may be combined with (A.6) to obtain the four-dimensional equations of motion

$$\begin{aligned}
R_{\mu\nu} &= \frac{n}{2} (\partial_\mu H \partial_\nu H + \partial_\mu G \partial_\nu G) + n \nabla_\mu \nabla_\nu H \\
&\quad + \frac{1}{2} e^{-n(H+G)} (F_{\mu\nu}^2 - \frac{1}{4} g_{\mu\nu} F^2) + \frac{1}{2} e^{-n(H-G)} (\tilde{F}_{\mu\nu}^2 - \frac{1}{4} g_{\mu\nu} \tilde{F}^2), \\
\Box H + n \partial^\mu H \partial_\mu H &= 2(n-1) e^{-H} \cosh G, \\
\Box G + n \partial^\mu H \partial_\mu G &= -\frac{1}{4} e^{-n(H+G)} F^2 + \frac{1}{4} e^{-n(H-G)} \tilde{F}^2 - 2(n-1) e^{-H} \sinh G.
\end{aligned} \tag{A.8}$$

The scalar equations were separated by taking appropriate linear combinations of the  $R_{ab}$  and  $R_{\tilde{a}\tilde{b}}$  equations. In addition, for  $n=1$  ( $D=6$ ), the reduction of the  $R_{a\tilde{a}}$  Einstein equation yields a constraint  $F_{\mu\nu} \tilde{F}^{\mu\nu} = 0$ . In general, this signifies an inconsistency in the reduction. However, so long as we satisfy this constraint, we are ensured that solutions to the effective four-dimensional theory may be lifted to solutions of the original six-dimensional theory.

The equations of motion, (A.5) and (A.8), may be derived from an effective four-dimensional Lagrangian

$$\begin{aligned}
e^{-1} \mathcal{L} &= e^{nH} \left[ R + \frac{1}{2} n(2n-1) \partial H^2 - \frac{1}{2} n \partial G^2 - \frac{1}{4} e^{-n(H+G)} F^2 \right. \\
&\quad \left. - \frac{1}{4} e^{-n(H-G)} \tilde{F}^2 + 2n(n-1) e^{-H} \cosh G \right].
\end{aligned} \tag{A.9}$$

If desired, this may be transformed into the Einstein frame by the Weyl transformation  $g_{\mu\nu} = e^{-nH} \tilde{g}_{\mu\nu}$ . The resulting Einstein frame action has the form

$$\begin{aligned}
\tilde{e}^{-1} \mathcal{L} &= \tilde{R} - \frac{1}{2} n(n+1) \partial H^2 - \frac{1}{2} n \partial G^2 - \frac{1}{4} e^{-nG} F^2 - \frac{1}{4} e^{nG} \tilde{F}^2 \\
&\quad + 2n(n-1) e^{-(n+1)H} \cosh G.
\end{aligned} \tag{A.10}$$

Note that the scalar fields are not canonically normalized. Nevertheless, we find it convenient to retain this convention, so as to avoid unpleasant factors of  $\sqrt{n}$  and  $\sqrt{n(n+1)}$ .

Finally, for the reductions we have considered, the form field  $F_{(n+2)}$  is taken to be self dual in  $D=10$  or  $6$ . Reducing the self-dual field follows the procedure given above, so long

as we impose the self-dual condition *after* obtaining the equations of motion, (A.2). In this case, the  $F^2$  term vanishes, and we are left with an Einstein equation of the form

$$R_{MN} = \frac{1}{2(n+1)!} (F^2)_{MN}. \quad (\text{A.11})$$

Note that, if canonical normalization is desired, we ought to include an additional factor of  $1/2$  in the field-strength term of the original Lagrangian, (A.1), in which case the right-hand side of (A.11) must also be multiplied by  $1/2$ . This is indeed what we do for the IIB theory. However, we forego this factor of  $1/2$  for the  $\mathcal{N} = (1,0)$  model in six dimensions. This choice of a non-canonically normalized 3-form  $H_{(3)}$  avoids  $\sqrt{2}$  factors in the supersymmetry transformation (3.3) of section 3 (and furthermore keeps canonical normalization in the case where the  $\mathcal{N} = (1,0)$  theory is coupled to a single tensor multiplet).

Regardless of normalization, for the self-dual case, we impose the condition  $\tilde{F}_{(2)} = - *_4 e^{-nG} F_{(2)}$  to eliminate  $\tilde{F}_{(2)}$  in favor of  $F_{(2)}$  in the reduced equations of motion. For canonical self-dual normalization, which incorporates the additional factor of  $1/2$  introduced above, this simply amounts to erasing all  $\tilde{F}$  terms from the expressions in (A.8). The resulting effective Lagrangians are identical to (A.9) and (A.10), except with the  $\tilde{F}^2$  terms removed. If we instead leave out the factor of  $1/2$ , the resulting  $F^2$  terms are twice as large (and the  $\tilde{F}^2$  terms are absent as usual). Here, we see the familiar feature that while  $\tilde{F}$  cannot be dualized in the Lagrangian, it is valid to do so in the equations of motion.

In addition, for self-duality in  $D = 6$ , the constraint  $F_{\mu\nu} \tilde{F}^{\mu\nu} = 0$  is replaced by  $F_{(2)} \wedge F_{(2)} = 0$ . Here it is clear that  $F_{(2)} \wedge F_{(2)}$  would ordinarily source an axionic field. However, by truncating away all axions, we can no longer allow such a source. Again, so long as we impose this constraint by hand, we will still be able to obtain solutions to the original six-dimensional model.

## B Integrability of the Killing spinor equations

Since we have found the somewhat surprising result that the Killing spinor equations resulting from both  $S^3 \times S^3$  compactification of IIB supergravity and  $S^1 \times S^1$  compactification of the  $\mathcal{N} = (1,0)$  theory have very similar forms, it is interesting to see how they can lead to different equations of motion, namely (A.8) with either  $n = 3$  or  $n = 1$ . In order to see how this works, we may consider the integrability of the Killing spinor equations, (4.1), repeated here as

$$\delta\psi_\mu = \mathcal{D}_\mu \epsilon, \quad \delta\chi_H = \Delta_H, \quad \delta\chi_G = \Delta_G, \quad (\text{B.1})$$

where

$$\mathcal{D}_\mu = \nabla_\mu + \frac{i}{16} e^{-\frac{n}{2}(H+G)} F_{\nu\lambda} \gamma^{\nu\lambda} \gamma_\mu,$$

$$\begin{aligned}
\Delta_H &= \gamma^\mu \partial_\mu H + e^{-\frac{1}{2}H} (\eta e^{-\frac{1}{2}G} - i\tilde{\eta}\gamma_5 e^{\frac{1}{2}G}), \\
\Delta_G &= \gamma^\mu \partial_\mu G + e^{-\frac{1}{2}H} (\eta e^{-\frac{1}{2}G} + i\tilde{\eta}\gamma_5 e^{\frac{1}{2}G}) - \frac{i}{4} e^{-\frac{n}{2}(H+G)} F_{\mu\nu} \gamma^{\mu\nu} \epsilon.
\end{aligned} \tag{B.2}$$

In the original theory (either in  $D = 10$  or  $6$ ), there is only one object to examine for first order integrability, namely  $[\mathcal{D}_M, \mathcal{D}_N]$ . However, viewed in the effective four-dimensional point of view, we may consider the various commutators of  $\mathcal{D}_\mu$ ,  $\Delta_H$  and  $\Delta_G$ . We begin with  $[\mathcal{D}_\mu, \mathcal{D}_\nu]$ . After straightforward although tedious manipulations, we find

$$\begin{aligned}
\gamma^\mu [\mathcal{D}_\mu, \mathcal{D}_\nu] &= \frac{1}{2} [R_{\nu\sigma} - \frac{n+1}{8} e^{-n(H+G)} (F^2_{\nu\sigma} - \frac{1}{4} g_{\mu\nu} F^2)] \\
&\quad - \frac{n}{2} (\partial_\nu H \partial_\sigma H + \partial_\nu G \partial_\sigma G) - n \nabla_\nu \nabla_\sigma H \gamma^\sigma \\
&\quad + \frac{i}{16} e^{-\frac{n}{2}(H+G)} \partial_{[\mu} F_{\lambda\sigma]} \gamma^{\mu\lambda\sigma} \gamma_\nu + \frac{i}{8} e^{-\frac{n}{2}(H+G)} \nabla^\mu (e^{-nG} F_{\mu\lambda}) \gamma^\lambda \gamma_\nu \\
&\quad + \frac{n}{2} [\mathcal{D}_\nu, \Delta_H] + \frac{n}{4} \partial_\nu H \Delta_H + \frac{n}{4} \partial_\nu G \Delta_G \\
&\quad - \frac{in}{32} e^{-\frac{n}{2}(H+G)} F_{\lambda\sigma} \gamma^{\lambda\sigma} \gamma_\nu (\Delta_H - \Delta_G).
\end{aligned} \tag{B.3}$$

Since the last two lines above vanish on Killing spinors, we see that this integrability yields the Einstein equation as well as Bianchi and equation of motion for  $F_{(2)}$ . In particular, if the latter two conditions are imposed on  $F_{(2)}$ , then the Einstein equation is guaranteed by supersymmetry. Note also that the Einstein equation of (A.8) is reproduced with proper dimension dependent ( $n = 1$  or  $3$ ) coefficients. This also shows the curious fact that, starting from an identical normalization of  $F_{(2)}$  in the supersymmetry variations, (B.2), one in fact obtains different normalizations in the bosonic equations involving  $F_{(2)}$ .

Turning to the  $[\mathcal{D}_\mu, \Delta_H]$  condition, we find

$$\begin{aligned}
\gamma^\mu [\mathcal{D}_\mu, \Delta_H] &= \square H + n \partial H^2 - (n-1) e^{-H} (\eta^2 e^{-G} + \tilde{\eta}^2 e^G) \\
&\quad + [-n \gamma^\mu \partial_\mu H - \frac{i}{8} e^{-\frac{n}{2}(H+G)} F_{\mu\nu} \gamma^{\mu\nu} \\
&\quad + (n - \frac{1}{2}) e^{-\frac{1}{2}H} (\eta e^{-\frac{1}{2}G} + i\tilde{\eta}\gamma_5 e^{\frac{1}{2}G})] \Delta_H \\
&\quad - \frac{1}{2} e^{-\frac{1}{2}H} (\eta e^{-\frac{1}{2}G} - i\tilde{\eta}\gamma_5 e^{\frac{1}{2}G}) \Delta_G,
\end{aligned} \tag{B.4}$$

which simply reproduces the  $H$  equation of motion in (A.8). In particular, for  $n \neq 1$  we are required to choose  $\eta^2 = \tilde{\eta}^2 = 1$ , while for  $n = 1$  these  $U(1)$  charges are irrelevant. This demonstrates that the identical bosonic equations are satisfied, regardless of the Kaluza-Klein charges carried by the Killing spinors.

At this stage, it is also worth noting that we may form the combination

$$[\gamma^\mu \partial_\mu H - e^{-\frac{1}{2}H} (\eta e^{-\frac{1}{2}G} + i\tilde{\eta}\gamma_5 e^{\frac{1}{2}G})] \Delta_H = \partial H^2 - e^{-H} (\eta^2 e^{-G} + \tilde{\eta}^2 e^G). \tag{B.5}$$

When acting on Killing spinors, this demonstrates that supersymmetry further imposes the condition

$$\partial H^2 = e^{-H} (\eta^2 e^{-G} + \tilde{\eta}^2 e^G). \tag{B.6}$$

Combining this with the equation of motion of  $H$  yields the simple expression

$$\square H + \partial H^2 = 0, \quad (\text{B.7})$$

which must be satisfied on supersymmetric backgrounds.

The final integrability condition we obtain is the one between  $\mathcal{D}_\mu$  and  $\Delta_G$ . In this case, we obtain

$$\begin{aligned} \gamma^\mu [\mathcal{D}_\mu, \Delta_G] = & \square G + n\partial H\partial G + \frac{n+1}{16}e^{-n(H+G)}F^2 - (n-1)e^{-H}(\eta^2 e^{-G} - \tilde{\eta}^2 e^G) \\ & + \frac{3-n}{32}e^{-n(H+G)}F_{\mu\nu}F_{\lambda\sigma}\gamma^{\mu\nu\lambda\sigma} \\ & - \frac{i}{4}e^{-\frac{n}{2}(H+G)}\partial_{[\mu}F_{\nu\lambda]}\gamma^{\mu\nu\lambda} - \frac{i}{2}e^{-\frac{n}{2}(H-G)}\nabla^\mu(e^{-nG}F_{\mu\nu})\gamma^\nu \\ & + \frac{1}{2}[-n\gamma^\mu\partial_\mu G + (n-1)e^{-\frac{1}{2}H}(\eta e^{-\frac{1}{2}G} - i\tilde{\eta}\gamma_5 e^{\frac{1}{2}G})]\Delta_H \\ & + \frac{1}{2}[-n\gamma^\mu\partial_\mu H + (n-1)e^{-\frac{1}{2}H}(\eta e^{-\frac{1}{2}G} + i\tilde{\eta}\gamma_5 e^{\frac{1}{2}G}) \\ & + \frac{i(n-1)}{4}e^{-\frac{n}{2}(H+G)}F_{\mu\nu}\gamma^{\mu\nu}]\Delta_G. \end{aligned} \quad (\text{B.8})$$

In addition to the  $G$  equation as well as Bianchi and equation of motion for  $F_{(2)}$ , we see that the  $F_{(2)} \wedge F_{(2)} = 0$  constraint shows up in this integrability condition provided  $n \neq 3$ . So, at least for the  $\mathcal{N} = (1, 0)$  theory, supersymmetry implies not just the equations of motion of (A.8), but also the  $F_{(2)} \wedge F_{(2)} = 0$  constraint.

More precisely, for partially broken supersymmetry, the Killing spinor equations often yield only linear combinations of the equations of motion. In this case, we see from (B.4) that both the  $H$  equation as well as the  $H$  constraint (B.7) are automatically satisfied independent of the rest of the fields. However, from (B.3) we see that the Einstein equation is only satisfied in combination with the  $F_{(2)}$  equations, and similarly from (B.8), that the  $G$  equation of motion is satisfied in combination with the  $F_{(2)}$  equations. We may conclude that, for obtaining supersymmetric backgrounds, it would be sufficient to satisfy the  $F_{(2)}$  Bianchi identity and equation of motion in addition to the Killing spinor equations themselves.

Finally, while the supersymmetry transformations (4.1) were only obtained for the cases  $n = 3$  and 1, they may nevertheless be formally extended to any value of  $n$ . Examination of the integrability conditions (B.3), (B.4) and (B.8) indicate consistency with a bosonic sector described by an effective Lagrangian

$$\begin{aligned} e^{-1}\mathcal{L} = & e^{nH}\left[R + \frac{1}{2}n(2n-1)\partial H^2 - \frac{1}{2}n\partial G^2 - \frac{n+1}{16}e^{-n(H+G)}F^2 \right. \\ & \left. + n(n-1)e^{-H}(\eta^2 e^{-G} + \tilde{\eta}^2 e^G)\right]. \end{aligned} \quad (\text{B.9})$$

For  $n \neq 3$  this system must be extended with the constraint  $F_{(2)} \wedge F_{(2)} = 0$ .

## C Differential identities for the spinor bilinears

The supersymmetric construction of [15–18] proceeds by postulating the existence of a Killing spinor  $\epsilon$  and then forming the tensors  $f_1$ ,  $f_2$ ,  $K_\mu$ ,  $L_\mu$  and  $Y_{\mu\nu}$  from spinor bilinears (4.2). The algebraic identities of interest were given in the text in (4.3). Here we tabulate the differential identities obtained by demanding that  $\epsilon$  solves the Killing spinor equations (4.1).

First, by assuming  $\delta\psi_\mu = 0$ , we may demonstrate that

$$\begin{aligned} \nabla_\mu f_1 &= \frac{1}{4}e^{-\frac{n}{2}(H+G)} * F_{\mu\nu} K^\nu, \\ \nabla_\mu f_2 &= -\frac{1}{4}e^{-\frac{n}{2}(H+G)} F_{\mu\nu} K^\nu, \\ \nabla_\mu K_\nu &= \frac{1}{4}e^{-\frac{n}{2}(H+G)} (f_2 F_{\mu\nu} - f_1 * F_{\mu\nu}), \\ \nabla_\mu L_\nu &= \frac{1}{4}e^{-\frac{n}{2}(H+G)} (\frac{1}{2}g_{\mu\nu} F_{\lambda\rho} Y^{\lambda\rho} - 2F_{(\mu}{}^\lambda Y_{\nu)\lambda}), \\ \nabla_\mu Y_{\nu\lambda} &= \frac{1}{4}e^{-\frac{n}{2}(H+G)} (2g_{\mu[\nu} F_{\lambda]\rho} L^\rho - 2F_{\mu[\nu} L_{\lambda]} + F_{\nu\lambda} L_\mu). \end{aligned} \quad (\text{C.1})$$

In particular, the equation for  $K_\mu$  indicates that  $K_{(\mu;\nu)} = 0$ , so that  $K^\mu$  is Killing. This is in fact a generic feature of constructing a Killing vector from Killing spinors.

In addition, the  $\delta\chi_H = 0$  condition allows us to derive the additional relations

$$\begin{aligned} K^\mu \partial_\mu H &= 0, & \eta f_2 &= -\tilde{\eta} e^G f_1, \\ L^\mu \partial_\mu H &= \eta e^{-\frac{1}{2}(H+G)} f_1 - \tilde{\eta} e^{-\frac{1}{2}(H-G)} f_2, \\ \eta e^{-\frac{1}{2}(H+G)} L_\mu &= f_1 \partial_\mu H, & \tilde{\eta} e^{-\frac{1}{2}(H-G)} L_\mu &= -f_2 \partial_\mu H, \\ \eta e^{-\frac{1}{2}(H+G)} K_\mu &= *Y_\mu{}^\nu \partial_\nu H, & \tilde{\eta} e^{-\frac{1}{2}(H-G)} K_\mu &= Y_\mu{}^\nu \partial_\nu H, \\ 2L_{[\mu} \partial_{\nu]} H &= 0, & 2K_{[\mu} \partial_{\nu]} H &= \eta e^{-\frac{1}{2}(H+G)} * Y_{\mu\nu} + \tilde{\eta} e^{-\frac{1}{2}(H-G)} Y_{\mu\nu}. \end{aligned} \quad (\text{C.2})$$

Similarly, the  $\delta\chi_G = 0$  condition yields the relations

$$\begin{aligned} K^\mu \partial_\mu G &= 0, & \frac{1}{4}e^{\frac{1-n}{2}(H+G)} F_{\mu\nu} * Y^{\mu\nu} &= \eta f_2 - \tilde{\eta} e^G f_1, \\ L^\mu \partial_\mu G &= \eta e^{-\frac{1}{2}(H+G)} f_1 + \tilde{\eta} e^{-\frac{1}{2}(H-G)} f_2 - \frac{1}{4}e^{-\frac{n}{2}(H+G)} F_{\mu\nu} Y^{\mu\nu}, \\ \eta e^{-\frac{1}{2}(H+G)} L_\mu &= f_1 \partial_\mu G + \frac{1}{2}e^{-\frac{n}{2}(H+G)} * F_{\mu\nu} K^\nu, \\ \tilde{\eta} e^{-\frac{1}{2}(H-G)} L_\mu &= f_2 \partial_\mu G + \frac{1}{2}e^{-\frac{n}{2}(H+G)} F_{\mu\nu} K^\nu, \\ \eta e^{-\frac{1}{2}(H+G)} K_\mu &= *Y_\mu{}^\nu \partial_\nu G + \frac{1}{2}e^{-\frac{n}{2}(H+G)} * F_{\mu\nu} L^\nu, \\ \tilde{\eta} e^{-\frac{1}{2}(H-G)} K_\mu &= -Y_\mu{}^\nu \partial_\nu G + \frac{1}{2}e^{-\frac{n}{2}(H+G)} F_{\mu\nu} L^\nu, \\ 2L_{[\mu} \partial_{\nu]} G &= 2e^{-\frac{n}{2}(H+G)} F_{[\mu}{}^\rho Y_{\nu]\rho}, \\ 2K_{[\mu} \partial_{\nu]} G &= \eta e^{-\frac{1}{2}(H+G)} * Y_{\mu\nu} - \tilde{\eta} e^{-\frac{1}{2}(H-G)} Y_{\mu\nu} - \frac{1}{2}e^{-\frac{n}{2}(H+G)} (f_1 * F_{\mu\nu} + f_2 F_{\mu\nu}). \end{aligned} \quad (\text{C.3})$$

Although the above identities are algebraic and not differential on the spinor bilinears, they originate from the supersymmetry variations along the internal directions of the Kaluza-Klein reduction. So in this sense, they form a generalized set of ‘differential identities’.

However, as they are only algebraic, they prove extremely useful in determining much of the geometry, as is evident from the analysis of [1].

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